Model Neurons: Neuroelectronics
(Part I)


- Basic electrical circuits.
- Passive membrane model.
- Single compartment model.
- Integrate-and-fire neurons.
- Hodgkin-Huxley model.
- Synaptic conductances.

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Electrical Circuits

- Ohm’s law:
  \[ V_R = I_R R, \]
  \( V: \) voltage, \( I: \) current, \( R: \) resistence.

- Charge across a capacitor:
  \[ CV_C = Q_C \]
  \[ C \frac{dV_C}{dt} = \frac{dQ_C}{dt} = I_C, \]
  \( V: \) voltage, \( Q: \) charge, \( I: \) current.

Electrical Circuits: Serial Resistors

Kirchhoff’s current law: At a node, all currents sum to zero (or, sum of incoming = sum of outgoing currents).

- Example C: at node next to \( V_2 \), \( I_1 = I_2 \). Thus:
  \[ V_1 - V_2 = I_1 R_1, \quad V_2 - 0 = I_2 R_2 \]
  \[ V_1 = I_1 (R_1 + R_2), \quad V_2 = I_2 R_2 = I_1 R_2 \]
  \[ V_2 = \frac{V_1 R_2}{R_1 + R_2}. \]

Electrical Circuits: Parallel Resistors

At the node next to \( V \), \( I_e = I_1 + I_2 \).

\[ I_1 = \frac{V}{R_1}, \quad I_2 = \frac{V}{R_2} \]
\[ I_e = \frac{V}{R_1} + \frac{V}{R_2} = \frac{R_1 + R_2}{R_1 R_2} V \]

Thus, total resistance of parallel resistors is \( \frac{R_1 R_2}{R_1 + R_2} \).
Resistor-Capacitor Circuit (I)

Case A: No external current source: \( I_R + I_C = 0 \).

\[ I_R + I_C = \frac{V - 0}{R} + C \frac{dV}{dt} = 0 \]

\[ C \frac{dV}{dt} = -\frac{V}{R} \]

which is a homogeneous linear differential equation, and the general solution is (straight-forward integration after separating the variables):

\[ V(t) = V(0) \exp(-t/RC) \].

The steady state of the membrane equation is:

\[ C \frac{dV}{dt} = \frac{E - V}{R} + I_e = 0, \]

\[ V = E + I_e R, \]

which we define as \( V_\infty = E + I_e R \), and the time constant is \( \tau = RC \), which gives the equation in the previous page:

\[ V(t) = V_\infty + (V(0) - V_\infty) \exp(-t/\tau). \]

For the solution, first get the general solution \( V_h \) for the homogeneous case and set \( V = V_h \cdot u \), where \( u \) is a dummy variable. Solve for \( V \).

Resistor-Capacitor Circuit (II)

Case B: With external current source: \( I_R + I_C = I_e \).

\[ I_R + I_C = \frac{V - E}{R} + C \frac{dV}{dt} = I_e \]

\[ C \frac{dV}{dt} = \frac{E - V}{R} + I_e \]

which is a nonhomogeneous linear differential equation, and the general solution is:

\[ V(t) = V_\infty + (V(0) - V_\infty) \exp(-t/\tau). \]

Single Compartment Model

- \( V \): membrane potential
- \( r_m \): specific membrane resistance
- \( c_m \): specific membrane capacitance
- \( I_e \): input current
- Conductance: reciprocal of resistance, denoted \( g \)

\[ \Delta V = I_e R_m \]

\[ R_m = r_m / A \]

\[ r_m \approx 1 \text{M}\Omega \cdot \text{mm}^2 \]

\[ Q = C_m V \]

\[ C_m = c_m A \]

\[ c_m \approx 10 \text{nF/mm}^2 \]
**Single Compartment Model: Circuit**

- Leakage current: \( i_L = \bar{g}_L (V - E_L) \).
- Membrane current: \( i_m = \sum_i g_i (V - E_i) \).
- Input current: \( I_e / A \).
- Current across capacitor: \( c_m \frac{dV}{dt} = I_C \).

**Integrate and Fire Models**

- Basically an RC circuit with the R-part serving as the leakage:
  \[
  c_m \frac{dV}{dt} = -\bar{g}_L (V - E_L) + \frac{I_e}{A}.
  \]
- Multiplying both sides with \( r_m \) gives
  \[
  (r_m = 1/\bar{g}_L, \tau_m = r_m c_m, R_m = r_m / A):
  \[
  \tau_m \frac{dV}{dt} = E_L - V + R_m I_e.
  \]
  When \( I_e = 0 \), steady state voltage becomes \( V = E_L \), which is the resting membrane potential (\( V_{rest} \)).
- When \( V \) reaches a threshold \( V_{th} \), generate a spike and reset the membrane potential to \( V_{rest} \).

**Integrate and Fire Models: Analytic Solution**

- Exact solution gives:
  \[
  V(t) = E_L + R_m I_e + (V(0) - E_L - R_m I_e) \exp(-t/\tau_m),
  \]
  which is the same as in page 7.
- \( V_{\infty} = E_L + R_m I_e \), and this value should be greater than the threshold \( V_{th} \) for the neuron to fire at all. Given a fixed \( E_L \) and \( R_m \), the only thing that can change \( V_{\infty} \) is then the input current \( I_e \).
- Given a constant input current \( I_e \) that allows spiking, the spiking frequency can be analytically calculated.
- First, calculate the time to first spike, when \( V(t) = V_{th} \) with \( V(0) = V_{rest} \), and solve for \( t \).
Integrate and Fire Models: Firing Rate

- The calculation comes out to:
  \[ t_{isi} = \tau_m \ln \left( \frac{R_m I_e + E_L - V_{rest}}{R_m I_e + E_L - V_{th}} \right). \]

- Since the neuron will fire every \( t_{isi} \) time units, this gives the “inter-spike interval” (or ISI).

- Thus, firing occurs with a period of \( t_{isi} \), and so the firing frequency is \( r_{isi} = 1/t_{isi} \).

- Note again that \( V_{th} < V_\infty = E_L + R_m I_e \) must hold. Otherwise, no spikes.

Integrate and Fire Model: Firing Rate

- Plot shows \( r_{isi} \) dependent on the input current (in INF vs. real data), and real neuron vs. INF firing.

- Without spike adaptation, INF fits the real data well (black dots).

- Spike adaptation means dynamic change in firing rate as a neuron keeps firing.

Integrate and Fire Model

- INF model with a fluctuating driving input is shown.

- The spikes (the long peaks) are shown just as a visualization, and they are not represented in the equation.

- Usually simple numerical integration is used for the simulation (use Taylor series expansion and drop higher-order terms):

  \[
  \tau_m \frac{\Delta V}{\Delta t} = E_L - V(t) + R_m I_e(t) \\
  \Delta V = \tau_m \frac{(E_L - V(t) + R_m I_e(t))}{\Delta t} \\
  V(t + \Delta t) = V(t) + \Delta V.
  \]